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2002 J. Phys. A: Math. Gen. 35 9245

(http://iopscience.iop.org/0305-4470/35/43/318)

J. Phys. A: Math. Gen. **35** (2002) 9245–9250 PII: S0305-4470(02)36611-3

Maximum principles and bounds in a class of fourth-order uniformly elliptic equations

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Received 3 May 2002, in final form 7 July 2002
Published 15 October 2002
Online at stacks.iop.org/JPhysA/35/9245

Abstract
This paper is devoted to a class of fourth-order uniformly elliptic equations posed by Schaefer in 1987. We obtain maximum principles for certain functions, which are defined on solutions of the elliptic equations. The principles are then used to deduce some bounds on important quantities in the physical problems of interest.

PACS numbers: 02.30.Jr, 02.60.Lj

1. Introduction

In 1972, Dunninger [1] first obtained a maximum principle for a fourth-order elliptic equation. This showed that any nonconstant solution u of

$$\triangle^2 u + cu = 0 \qquad c > 0 \quad \text{in } \Omega \subset R^n$$
$$\triangle u = 0 \quad \text{on } \partial\Omega$$

satisfies the inequality

$$|u(x)| \leqslant |u(x_0)| \qquad x \in \overline{\Omega}$$

for some point x_0 on the boundary $\partial\Omega$ of Ω, where $\triangle$ is the Laplacian, $\triangle^2 = \triangle(\triangle)$.

Later, several authors extended this work (see, for example, [2–5] etc). In 1987, Schaefer [6] found that the function $V(x)$ of the solution $u \in C^4(\Omega)$ of

$$\triangle^2 u + g(x, u, \triangle u) + p(x) f(u) = 0 \quad \text{in } \Omega \tag{$*$}$$

under some assumptions satisfies the inequality

$$V(x) \leqslant V(x_0) \qquad x \in \overline{\Omega}$$

for some point x_0 on the boundary $\partial\Omega$ of Ω, where

$$V(x) := u_{,i} u_{,i} + \gamma (\triangle u)^2 + 2\gamma p(x) \int_0^u f(s)\, \mathrm{d}s.$$

In [6], Schaefer posed an open problem of whether there are analogous results when $\triangle u$ is replaced by uniformly elliptic operator $Lu = a_{ij}u_{,ij}$ in equation ($*$). For the following equation

$$L^2u + g(x, u, Lu) + p(x)f(u) = 0 \quad \text{in } \Omega$$

the usual methods are infeasible, owing to the higher order and nonconstant coefficients a_{ij}, so the problem has so far remained open.

In this paper, the problem is resolved completely. Here, the function $V(x)$ takes the form

$$V(x) := u_{,i}u_{,i} + \gamma(Lu)^2 + 2\gamma p(x)\int_0^u f(s)\,\mathrm{d}s.$$

We show that $V(x)$ satisfies the maximum principle. In the process of computing LV, it is difficult to deal with the term of $a_{ij}u_{,ik}u_{,jk} - a_{ij,k}u_{,k}u_{,ij}$. Compared with the case $a_{ij} = \delta_{ij}$ in [1], it is more complicated in the case $a_{ij} \neq \delta_{ij}$. However, the difficulty is properly solved by using the matrix technique. Our main results are presented in theorems 1 and 2 of section 2. In section 3, we deduce certain bounds for quantities of interest in some physical problems, such as, the solution of the equation, gradient of the solution and so on.

For simplicity, we use the summation convention and denote partial derivatives $\frac{\partial u}{\partial x_i}$ by $u_{,i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ by $u_{,ij}$.

2. Maximum principles

Consider the equation

$$L^2u + g(x, u, Lu) + p(x)f(u) = 0 \quad \text{in } \Omega \tag{1}$$

where Ω is a nonempty bounded open domain in R^n, $Lu := a_{ij}u_{,ij}$ is a uniformly elliptic operator and $a_{ij}(x) = a_{ji}(x)$, and $a_{ij} \in C(\overline{\Omega})$. We assume that the coefficient functions $p > 0$, $p \in C(\overline{\Omega})$ and $g \in C(R \times R \times \overline{\Omega})$, and functions f and g satisfy the requirements

$$f'(s) > 0 \qquad sg(x, t, s) \leqslant 0. \tag{2}$$

2.1. Function without gradient term

Letting $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution of equation (2.1), we define the function

$$V(x) := (Lu)^2 + 2p(x)F(u) \tag{3}$$

where $F(u) = \int_0^u f(t)\,\mathrm{d}t$.

Theorem 1. *Letting $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution of equation (1), where $p(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ and $f(u) \in C^1(R)$, $p(x) > 0$ and $Lp^{-1} \leqslant 0$, $f'(u) > 0$, $F(u) \geqslant 0$ and $2F(u)F''(u) - [F'(u)]^2 \geqslant 0$, then g satisfies equation (2) and $V(x)$ defined in equation (3) satisfies*

$$V(x) \leqslant V(x_0) \qquad x \in \overline{\Omega} \tag{4}$$

where x_0 is some point on $\partial\Omega$.

Proof. By a straightforward computation of the uniformly elliptic operator, we have

$$\begin{aligned} LV = {} & 2a_{kl}(Lu)_{,k}(Lu)_{,l} + 2(Lu)(L^2u) + 2(Lp)F + 2f(a_{kl}p_{,k}u_{,l} + a_{kl}u_{,k}p_{,l}) \\ & + 2pf(Lu) + 2pf'(a_{kl}u_{,k}u_{,l}). \end{aligned} \tag{5}$$

Furthermore, from equations (1) and (5) we can write

$$LV = 2a_{kl}(Lu)_{,k}(Lu)_{,l} - 2(Lu)g(x,u,Lu) + 2pf'\left[a_{kl}\left(u_{,k} + \frac{fp_{,k}}{pf'}\right)\left(u_{,l} + \frac{fp_{,l}}{pf'}\right)\right] + 2(Lp)F - \frac{2f^2}{pf'}(a_{kl}p_{,k}p_{,l}). \tag{6}$$

From $p > 0$, $Lp^{-1} \leqslant 0$, i.e. $p(Lp) - 2a_{kl}p_{,k}p_{,l} \geqslant 0$, it follows that

$$(Lp)F - \frac{f^2}{pf'}(a_{kl}p_{,k}p_{,l}) \geqslant \frac{a_{kl}p_{,k}p_{,l}}{pf'}(2FF'' - F'^2). \tag{7}$$

Using equations (6) and (7) we have

$$LV \geqslant 2\left[a_{kl}(Lu)_{,k}(Lu)_{,l} - (Lu)g(x,u,Lu) + pf'a_{kl}\left(u_{,k} + \frac{fp_{,k}}{pf'}\right)\left(u_{,l} + \frac{fp_{,l}}{pf'}\right) + \frac{a_{kl}p_{,k}p_{,l}}{pf'}(2FF'' - F'^2)\right]. \tag{8}$$

The uniform ellipticity of the operator L and the assumptions of theorem 1 guarantee that the right-hand side of equation (8) is non-negative. Thus, $LV \geqslant 0$ and the conclusion (4) follows immediately from Hopf's first maximum principle. □

2.2. *Function with gradient term*

Letting $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution of equation (1), we define the function

$$V(x) := |\nabla u|^2 + \gamma(Lu)^2 + 2\gamma p(x)F(u) \tag{9}$$

where γ is a positive constant to be chosen and $F(u)$ is a primitive function of the function f, i.e. $F(u) = \int_0^u f(t)\,\mathrm{d}t$.

We deduce the maximum principles of the functional $V(x)$ now given by equation (9).

Analogous with the proof of theorem 1, by a straightforward calculation, we have

$$LV = 2a_{ij}u_{,ik}u_{,jk} + 2u_{,k}(Lu_{,k}) + 2\gamma[a_{ij}(Lu)_{,i}(Lu)_{,j} - (Lu)g(x,u,Lu) + F(Lp) + fa_{ij}(u_{,i}p_{,j} + p_{,i}u_{,j}) + pf'a_{ij}u_{,i}u_{,j}]. \tag{10}$$

Now let α be any constant satisfying $p(x) \geqslant p_0 > \alpha > 0$ and $P = P(x) = p(x) - \alpha$. By changing the form of $2u_{,k}(Lu_{,k})$, we have

$$LV = (2a_{ij}u_{,ik}u_{,jk} - 2u_{,k}a_{ij,k}u_{,ij}) + 2u_{,k}(Lu)_{,k} + 2\gamma[a_{ij}(Lu)_{,i}(Lu)_{,j} - (Lu)g(x,u,Lu) + \alpha f'a_{ij}u_{,i}u_{,j} + \{F(LP) + fa_{ij}(u_{,i}P_{,j} + P_{,i}u_{,j}) + Pf'a_{ij}u_{,i}u_{,j}\}]. \tag{11}$$

First, we handle the first item on the right-hand side of equation (11). Denoting by (A_{ij}) the matrix which is the inverse of positive definite matrix (a_{ij}), we see that, for arbitrary $n \times n$ matrix (S_{pk}),

$$a_{ij}\left(u_{,ik} + \frac{A_{ip}S_{pk}}{2}\right)\left(u_{,jk} + \frac{A_{jq}S_{qk}}{2}\right) \geqslant 0.$$

Therefore, the inequality

$$S_{jk}u_{,jk} \geqslant -a_{ij}u_{,ik}u_{,jk} - \frac{A_{pq}S_{pk}S_{qk}}{4} \tag{12}$$

is valid for any matrix (S_{jk}). By choosing $S_{ij} = -a_{ij,k}u_{,k}$ in equation (12) we obtain

$$a_{ij}u_{,ik}u_{,jk} - a_{ij,k}u_{,k}u_{,ij} \geqslant -\frac{A_{pq}a_{pk,i}a_{qk,j}u_{,i}u_{,j}}{4}. \tag{13}$$

Clearly, the second term in equation (11) can be turned into

$$2u_{,k}(Lu)_{,k} = |u_{,k} + (Lu)_{,k}|^2 - u_{,k}u_{,k} - (Lu)_{,k}(Lu)_{,k}. \tag{14}$$

We finally complete the square in the brackets in equation (11), i.e.

$$\begin{aligned} &F(LP) + fa_{ij}(u_{,i}P_{,j} + P_{,i}u_{,j}) + Pf'a_{ij}u_{,i}u_{,j} \\ &\quad = F(LP) + Pf'a_{ij}\left(u_{,i} + \frac{fP_{,i}}{Pf'}\right)\left(u_{,j} + \frac{fP_{,j}}{Pf'}\right) - \frac{f^2}{Pf'}a_{ij}P_{,i}P_{,j}. \end{aligned}$$

Now let f and F satisfy the requirements

$$f'(u) \geqslant \beta > 0 \qquad F(u) > 0 \qquad 2F(u)F''(u) - [F'(u)]^2 \geqslant 0 \tag{15}$$

$P(x)$ satisfies the requirement

$$P(LP) - 2a_{ij}P_{,i}P_{,j} \geqslant 0 \tag{16}$$

which is equivalent to $LP^{-1} \leqslant 0$. Under these additional assumptions, we can write

$$\begin{aligned} F(LP) + fa_{ij}(u_{,i}P_{,j} + P_{,i}u_{,j}) + Pf'a_{ij}u_{,i}u_{,j} &\geqslant Pf'a_{ij}\left(u_{,i} + \frac{fP_{,i}}{Pf'}\right)\left(u_{,j} + \frac{fP_{,j}}{Pf'}\right) \\ &\quad + \frac{a_{ij}P_{,i}P_{,j}}{Pf'}(2FF'' - F'^2) \geqslant 0. \end{aligned} \tag{17}$$

Consequently, we obtain by equations (11), (13), (14) and (17)

$$\begin{aligned} LV \geqslant\ & (2\gamma a_{ij} - \delta_{ij})(Lu)_{,i}(Lu)_{,j} + \left(2\alpha\beta\gamma a_{ij} - \delta_{ij} - \frac{A_{pq}a_{pm,i}a_{qm,j}}{2}\right)u_{,i}u_{,j} \\ & + 2\gamma Pf'a_{ij}\left(u_{,i} + \frac{fP_{,i}}{Pf'}\right)\left(u_{,j} + \frac{fP_{,j}}{Pf'}\right) + 2\gamma\left(\frac{a_{ij}P_{,i}P_{,j}}{Pf'}\right)(2FF'' - F'^2) \\ & + |u_{,k} + (Lu)_{,k}|^2 - 2\gamma(Lu)g(x, u, Lu). \end{aligned} \tag{18}$$

Since (a_{ij}) is uniformly positive definite, the right-hand side of equation (18) is positive for a sufficiently large value of γ. This value of γ depends only on constants α and β, and the coefficients a_{ij} and their first derivatives. Hence $V(x)$ is subharmonic. In summary, we have the following result.

Theorem 2. *Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution of equation (1). If $a_{ij}(x) \in C^1(\overline{\Omega})$, $p(x) \in C^4(\Omega)$, $f(u) \in C^1(R)$ and the functions p, f and g satisfy the requirements*

(i) $p(x) \geqslant p_0 > \alpha > 0$ for some constant α, and $LP^{-1} \leqslant 0$, where $P(x) = p(x) - \alpha$;
(ii) $f(0) = 0$, $f'(u) \geqslant \beta > 0$ for some constant β, and $2F(u)F''(u) - [F'(u)]^2 \geqslant 0$;
(iii) $sg(x, t, s) \leqslant 0$;

then there is a positive constant γ depending only on α, β, a_{ij} and $a_{ij,m}$ such that the function $V(x)$ given by (9) satisfies the maximum principle in Ω, i.e.

$$V(x) \leqslant V(x_0) \qquad x \in \overline{\Omega}$$

where x_0 is some point on $\partial\Omega$.

3. Applications: bounds for $|\nabla u|$, u and uniformly elliptic operator Lu

By theorem 1 one can easily obtain bounds for the uniformly elliptic operator Lu and the solution of (1).

Corollary 1. *Let u be a sufficiently smooth solution of equation (1). Suppose that the assumptions of theorem 1 are satisfied and F is an even function. Then*

$$|u(x)| \leqslant |u(x_0)|$$

for all $x \in \Omega$ and some point $x_0 \in \partial\Omega$ provided $Lu = 0$ on $\partial\Omega$ and $p(x_0) \leqslant p(x)$.

Proof. In theorem 1 the subharmonic function $V(x)$ obtains its maximum at some point, say x_0 on the boundary of Ω. Thus it follows that

$$(Lu(x))^2 + 2p(x)\int_0^{u(x)} f(t)\,\mathrm{d}t \leqslant (Lu(x_0))^2 + 2p(x_0)\int_0^{u(x_0)} f(t)\,\mathrm{d}t.$$

However, since $Lu(x_0) = 0$, it yields

$$2p(x)\int_0^{u(x)} f(t)\,\mathrm{d}t \leqslant (Lu(x))^2 + 2p(x)\int_0^{u(x)} f(t)\,\mathrm{d}t \leqslant 2p(x_0)\int_0^{u(x_0)} f(t)\,\mathrm{d}t.$$

In addition to $p(x) \geqslant p(x_0) > 0$, we have

$$|u(x)| \leqslant |u(x_0)|$$

for all $x \in \Omega$, which proves the desired result. □

Corollary 2. *Let u be a sufficiently smooth solution of equation (1). Suppose that the assumptions of theorem 1 are satisfied, then*

$$|Lu(x)| \leqslant |Lu(x_0)|$$

for all $x \in \Omega$ and for some point $x_0 \in \partial\Omega$ provided $u = 0$ on $\partial\Omega$.

Proof. It follows from theorem 1 that at some point $x_0 \in \partial\Omega$

$$(Lu(x))^2 + 2p(x)F(u(x)) \leqslant (Lu(x_0))^2 + 2p(x_0)F(u(x_0)).$$

However, $u(x_0) = 0$, hence $F(u(x_0)) = 0$, which yields

$$(Lu(x))^2 \leqslant (Lu(x))^2 + 2p(x)F(u(x)) \leqslant (Lu(x_0))^2$$

i.e.

$$|Lu(x)| \leqslant |Lu(x_0)|$$

for all $x \in \Omega$. This proves the desired result. □

With analogy to theorem 1, one can also obtain bounds for the uniformly elliptic operator and the solution of equation (1) from theorem 2. In particular, we can obtain the bound for the gradient of the solution (1), and the bound does not depend on the value of solution at the point in question.

Corollary 3. *Let u be a sufficiently smooth solution of equation (1). Suppose that the assumptions of theorem 2 are satisfied, then there is a positive constant $\gamma = \gamma(\alpha, \beta, a_{ij}, a_{ij,k})$ such that*

$$|\nabla u(x)|^2 \leqslant |\nabla u(x_0)|^2 + \gamma[Lu(x_0)]^2 + 2\gamma p(x_0)\int_0^{u(x_0)} f(t)\,\mathrm{d}t$$

for all $x \in \Omega$, and for some point $x_0 \in \partial\Omega$.

The proof of this corollary is easily achieved by theorem 2.

We have only briefly indicated how theorems 1 and 2 in section 2 can be utilized to obtain point-wise bounds. Many other similar applications of the principles in section 2 can be found in [5] and [7].

Acknowledgments

We would like to thank Professor Li-He Wang and Professor Ming-Qi Yu for their helpful discussion. We should also like to thank the referees for their advice on amendments. This work is partially supported by the Open Project of the Defense Key Laboratory of Science and Technology (00JS76.4.2JW0810), the Talent Teacher Programme of the Chinese Ministry of Education, and the Open Project of the State Key Laboratory of Multiple Flows (JW0810).

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